Student Notes

Student Study Session Topic: Differential Equations

Differential equations are tested every year. The actual solving of the differential equation is usually the main part of a free-response question (5 or 6 points). The rest of the question may be a slope field, a tangent line approximation or some other related topic. BC students may also be asked to approximate a value using Euler's Method. Parts of the BC questions are often suitable for AB students and are used to compute to the AB sub-score of the BC exam.

There may also be 2 or 3 multiple-choice problems each year.

- **What you should be able to do:**
  - Find the *general solution* of a differential equation using the method of *separation of variables* (this is the *only* method tested).
  - Find a *particular solution* using the initial condition to evaluate the constant of integration – initial value problem (IVP)
  - Understand that the proposed solution of a differential equation is a function (not a number) and if it and its derivative(s) are substituted into the given differential equation the resulting equation is true. This may be part of doing the problem even if solving the differential equation is not required (see 2002 BC 5(c))
  - Growth-decay problems.
  - Slope fields;
    - Draw a slope field by hand.
    - Identify the differential equation for a given slope field.
    - Sketch a *particular solution* on a given slope field.
    - Interpret a slope field.

  For BC only:

  - Use Euler's Method to approximate a solution.
  - Use the method of integration by parts or partial fractions to find the antiderivative after separating the variables.
  - Understand the logistic growth model, its asymptotes, meaning, etc.

Try these free-response questions as instructed by your presenter.

2005 AB 6

(a) Slope field

(b) Tangent line approximation

(c) Notice how the points are earned- most for knowing what to do. Also notice the negative sign in front of the radical. This is determined by the initial condition.

2006 AB 5

(a) Slope field

(b) The domain is the largest *continuous* interval that contains the initial condition point and for which the solution satisfies the differential equation. In other words, the solution cannot cross a vertical asymptote or go through a point where the function is undefined (in this problem, (0, -1)).

2002 BC 5

(a) Slope field- the graph must contain the initial condition point and extend to the edges of the graph.

(b) BC: Euler's method

(c) Tests the idea that the solution musts check in the differential equation. Guess (look at the graph) and check by substituting or use the solution with *b* and substitute into the differential equation to solve for *b*.

(d) Second derivative test.

2008 AB 5

(b) There is a difference of opinion of just when to substitute the original condition: right after the antidifferentiation and +C, or later. The solution on the scoring rubric opts for the latter, which allows *k* to be negative and avoid some problems with removing the absolute value signs. The other possibility is to substitute immediately after line 2. That solution would then become:

$$|0-1| = e^{-\frac{1}{2}+C}$$

$$1 = e^{-\frac{1}{2}+C} \Rightarrow \ln 1 = C - \frac{1}{2} \Rightarrow C = \frac{1}{2}$$

$$|y-1| = e^{-\frac{1}{x}+\frac{1}{2}} = e^{\left(\frac{1}{2}-\frac{1}{x}\right)}$$

Near the initial condition point (2, 0), $y - 1 < 0$ so $|y - 1| = -(y - 1) = 1 - y$. Thus,

$$1 - y = e^{\left(\frac{1}{2}-\frac{1}{x}\right)}$$

$$y = 1 - e^{\left(\frac{1}{2}-\frac{1}{x}\right)},\ x > 0$$

You must be careful with the negative signs; otherwise you will get a slightly different and slightly wrong answer!

Now try the multiple-choice questions:

1-7 are for AB and 8-11 are for BC.

Answers: 1. a 2. b 3. b 4. e 5. d 6. c 7. e 8. c 9. b 10. b 11. e

WATCH and LISTEN to the multiple-choice questions being solved:

Go to http://tinyurl.com/NMSI-Math-8b

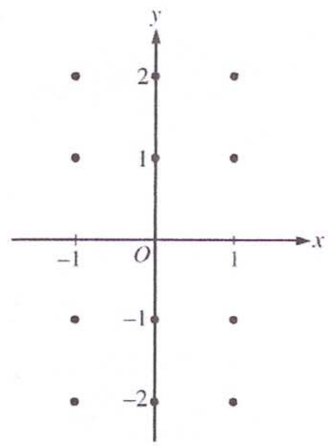
Click on the "Full Screen" arrow. Then, click anywhere on the page to see and hear from that point on.

Click anywhere to back anytime.

2005 AP® CALCULUS AB FREE-RESPONSE QUESTIONS

6. Consider the differential equation $\frac{dy}{dx} = -\frac{2x}{y}$.

   (a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.
   (Note: Use the axes provided in the pink test booklet.)



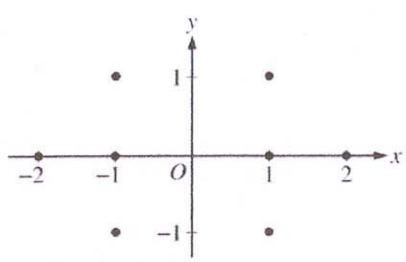
   (b) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = -1$. Write an equation for the line tangent to the graph of f at $(1, -1)$ and use it to approximate $f(1.1)$.

   (c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(1) = -1$.

2006 AP® CALCULUS AB FREE-RESPONSE QUESTIONS

5. Consider the differential equation $\frac{dy}{dx} = \frac{1+y}{x}$, where $x \neq 0$.

   (a) On the axes provided, sketch a slope field for the given differential equation at the eight points indicated.
   (Note: Use the axes provided in the pink exam booklet.)

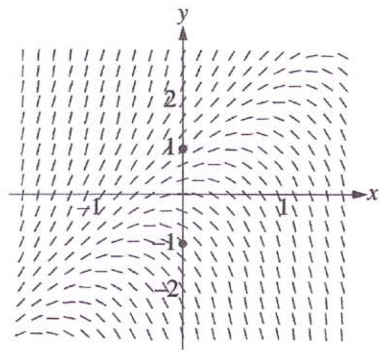


   (b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(-1) = 1$ and state its domain.

2002 AP® CALCULUS BC FREE-RESPONSE QUESTIONS

5. Consider the differential equation $\frac{dy}{dx} = 2y - 4x$.

   (a) The slope field for the given differential equation is provided. Sketch the solution curve that passes through the point $(0, 1)$ and sketch the solution curve that passes through the point $(0, -1)$.
   (Note: Use the slope field provided in the pink test booklet.)



   (b) Let f be the function that satisfies the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with a step size of 0.1, to approximate $f(0.2)$. Show the work that leads to your answer.

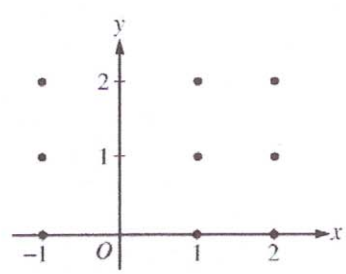
   (c) Find the value of b for which $y = 2x + b$ is a solution to the given differential equation. Justify your answer.

   (d) Let g be the function that satisfies the given differential equation with the initial condition $g(0) = 0$. Does the graph of g have a local extremum at the point $(0, 0)$? If so, is the point a local maximum or a local minimum? Justify your answer.

2008 AP® CALCULUS AB FREE-RESPONSE QUESTIONS

5. Consider the differential equation $\frac{dy}{dx} = \frac{y-1}{x^2}$, where $x \neq 0$.

   (a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.
   (Note: Use the axes provided in the exam booklet.)



   (b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(2) = 0$.

   (c) For the particular solution $y = f(x)$ described in part (b), find $\lim_{x \to \infty} f(x)$.

Multiple Choice

Identify the choice that best completes the statement or answers the question. There is no penalty for guessing.

1. Population y grows according to the equation $\frac{dy}{dx} = ky$ where k is a constant and t is measured in years. If the population doubles every 10 years, then the value of k is

   a) 0.069 b) 0.200 c) 0.301 d) 3.322 e) 5.000

2. If $\frac{dy}{dt} = ky$ and k is a nonzero constant, then y could be

   a) $2e^{ky}$ b) $2e^{kt}$ c) $e^{kt} + 3$ d) $kty + 5$ e) $\frac{1}{2}ky^2 + \frac{1}{2}$

3. If $\frac{dy}{dx} = \sin x \cos^2 x$ and if y = 0 when $x = \frac{\pi}{2}$, what is the value of y when x = 0?

   a) – 1 b) $-\frac{1}{3}$ c) 0 d) $\frac{1}{3}$ e) 1

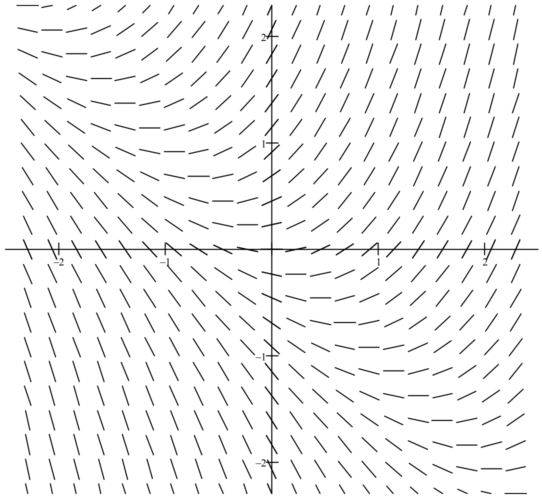
4. The rate of change of the volume, V, of water is a tank with respect to time, t, is directly proportional to the square root of the volume. Which of the following is a differential equation that describes this relationship?

   a) $V(t) = k\sqrt{t}$ b) $V(t) = k\sqrt{V}$ c) $\frac{dV}{dt} = k\sqrt{t}$ d) $\frac{dV}{dt} = \frac{k}{\sqrt{V}}$ e) $\frac{dV}{dt} = k\sqrt{V}$

5. A curve has slope $2x + 3$ at each point (x, y) on the curve. Which of the following is an equation for this curve if it passes through the point (1, 2)?

   a) $y = 5x - 3$ b) $y = x^2 + 1$ c) $y = x^2 + 3x$ d) $y = x^2 + 3x - 2$ *e)* $y = x^2 + 3x - 3$

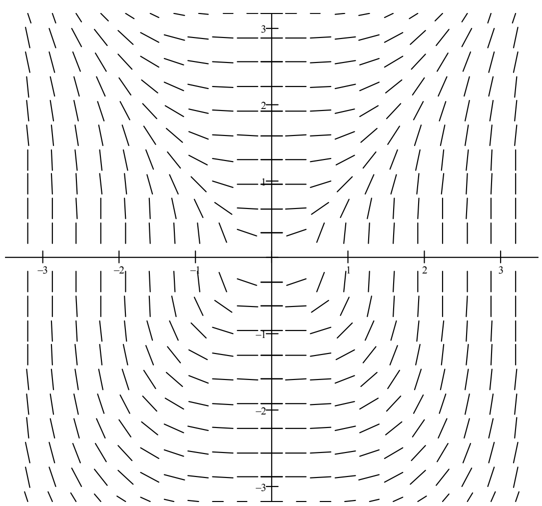
6.



Shown above is a slope field for which of the following differential equations?

a) $\dfrac{dy}{dx} = 1 + x$ b) $\dfrac{dy}{dx} = x^2$ c) $\dfrac{dy}{dx} = x + y$ d) $\dfrac{dy}{dx} = \dfrac{x}{y}$ e) $\dfrac{dy}{dx} = \ln y$

7.



Shown above is a slope field for which of the following differential equations?

a) $\dfrac{dy}{dx} = \dfrac{x}{y}$ b) $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$ c) $\dfrac{dy}{dx} = \dfrac{x^3}{y}$ d) $\dfrac{dy}{dx} = \dfrac{x^2}{y}$ e) $\dfrac{dy}{dx} = \dfrac{x^3}{y^2}$

8. (BC) Let $y=f(x)$ be the solution to the differential equation $\frac{dy}{dx}=x+y$ with the initial condition $f(1)=2$. What is the approximation for $f(2)$ if Euler's method is used, starting at $x=1$ with a step size of 0.5?

a) 3 b) 5 c) 6 d) 10 e) 12

9. (BC) The length of a curve from $x=1$ to $x=4$ is given by $\int_1^4 \sqrt{1+9x^4}\,dx$. If the curve contains the point (1, 6), which of the following could be an equation for this curve?

a) $y=3+3x^2$ b) $y=5+x^3$ c) $y=6+x^3$ d) $y=6-x^3$ e) $y=\frac{16}{5}+x+\frac{9}{5}x^5$

10. (BC) The number of moose in a national park is modeled by the function M that satisfies the differential equation $\frac{dM}{dt}=0.6M\left(1-\frac{M}{200}\right)$, where t is the time in years and $M(0)=50$. What is $\lim\limits_{t\to\infty} M(t)$?

a) 50 b) 200 c) 500 d) 1000 e) 2000

11. (BC) If $\frac{dy}{dx}=(1+\ln x)y$ and if $y=1$ when $x=1$, then $y=$

a) $e^{\frac{x^2-1}{x^2}}$ b) $1-\ln x$ c) $\ln x$ d) $e^{2x+x\ln x-2}$ e) $e^{x\ln x}$